Max Poole Practice exam 2

!= means “does not equal”

=== means congruent (or type conversion equality if you like javascript)

1. Let Fi be the ith Fibonacci number where F1 = 1 and F2 = 1 and Fi = F1 + F2. Prove that the sum of Fi from 1 to n (where n is any positive integer) is equal to Fn+2 -1.
2. Let sets R, S, and T be defined as follows:

R = {x ∈ Z | x is divisible by 2}

S = {y ∈ Z| y is divisible by 3}

T = {z ∈ Z| z is divisible by 6}

a. Is R ⊆ T?

b. Is T ⊆ R?

c. Is T ⊆ S?

3. Given Z, Q, R, and N as the set of integers, rational numbers, real numbers, and natural numbers respectively, tell which of the following relationships are true.

a) Z+ ⊆ Q

b) Z− ∩ Z+ = ∅

c) Z ∪ Q = Z

d) N ∪ (R – Q) = ∅

4) Let Ci = {i,−i } for all nonnegative integers i .

a. U4i=0 Ci = ?

b. ∩4 i=0 Ci =?

c. Are C0,C1,C2, . . . mutually disjoint?

5) Prove using an element argument that for all sets A and B,   
 (A ∩ B) ∪ (A ∩ Bc) = A.

6) For A = {1,2,3}, B = {2,3,4}, C = {3,4,5} find

(A X B) ∪ (A X C).

7) Given the relation on A = [1,2] and B = [1,2,4,5] defined by

{(1,1),(2,2),(1,5),(1,2),(2,4),(1,4)}

1. Find the inverse relation
2. Is this relation reflexive?
3. Is this relation symmetric?
4. Is this relation antisymmetric?
5. Is this relation transitive?
6. Is this relation an equivalence relation?
7. Is this relation a partial order relation?
8. Could you make an equivalence relation using A and B?

8) Find numbers congruent to each of the following.

a) 500500 mod 5

b) 392 mod 6

c) 166 mod 3

9) Give a surjection from the set of real numbers onto the set A = {2}. If this is not possible, say why.

10) State whether the following functions are injective (one-to-one), surjective (onto), or both (bijective or one to one correspondence). Given an example if something is not injective or surjective.

a) Function F: R -> R F(x) = 2x

b) Function G: R-> R G(x) = 2x when x != 1 or

3 when x = 1

c) Function H:N->R H(x) = 2x

11) For g: R-> R g = sin(x) and f: R-> R f = π\*x, calculate the following.

a) g ◦ f(1/2)

b) f ◦ g(0)

12) Make a surjection from N->R or say why it is impossible. (Hint: What are the cardinalities of these two sets?)

13) Solve these logarithm problems (without a calculator)

a) log381

b) ln e2

Solutions

1. Let Fi be the ith Fibonacci number where F1 = 1 and F2 = 1 and Fi = F1 + F2. Prove that the sum of Fi from 1 to n (where n is any positive integer) is equal to Fn+2 -1.

**Let P(n) be “The sum Fi from 1 to n = Fn+2 -1.”**

**P(1) says that F1 = F3 -1**

**F1 = 1**

**F3 = F2 + F1 = 1 + 1 = 2**

**1 = 2 -1 so P(1) is true**

**P(2) says that F1 + F2 = F4 – 1**

**F1 + F2 = 1 + 1 = 2**

**F4 = F3 + F2 = 2 + 1 = 3**

**2 = 3 -1 so P(2) is true**

**Inductive Hypothesis: Now assume that for some arbitrary but discrete integer k that P(n) is true for all integers less than or equal to k.**

**Inductive Step: Then I can prove that P(k+1) is true.**

**P(k+1) says that F1 + F2 + … + FK + F(K+1) = F(K+3) – 1**

**P(K) says that F1 + F2 + … + FK = F(K+2) – 1.**

**If we add F(K+1) to both sides of the equation given by P(K) we get**

**F1 + F2 + .. + FK + F(K+1) = F(K+2) + F(K+1) – 1**

**Since F(K+3) = F(K+2) + F(K+1) we get that**

**F1 + F2 + … + FK + F(K+1) = F(K+3) – 1**

**Conclusion:**

**Since P(n) is true for n = 1 and n =2, and since when P(n) is true for all n less than or equal to some fixed integer k we know that P(k+1) is true, P(n) is true for all positive integers by strong induction.**

1. **Let sets R, S, and** T be defined as follows:

R = {x ∈ Z | x is divisible by 2}

S = {y ∈ Z| y is divisible by 3}

T = {z ∈ Z| z is divisible by 6}

a. Is R ⊆ T? **No, 4 is in R but not in T**

b. Is T ⊆ R? **Yes, every number divisible by 6 is also divisible by 2**

c. Is T ⊆ S? **Yes, every number divisible by 6 is also divisible by 3**

3. Given Z, Q, R, and N as the set of integers, rational numbers, real numbers, and natural numbers respectively, tell which of the following relationships are true.

a) Z+ ⊆ Q **True, every positive integer is also a rational number**

b) Z− ∩ Z+ = ∅ **True, no integer is both positive and negative**

c) Z ∪ Q = Z **False, there are some rational numbers that are not integers**

d) N ∪ (R – Q) = ∅ **False, the union of these two sets contains both the irrationals and naturals.**

4) Let Ci = {i,−i } for all nonnegative integers i .

a. U4i=0 Ci = **Union of [0], [-1,1], [-2,2], [-3,3], [-4,4] = [-4,4]**

b. ∩4 i=0 Ci =? **Intersection of [0], [-1,1], [-2,2], [-3,3], [-4,4] = [0]**

c. Are C0,C1,C2, . . . mutually disjoint? **No, given positive integers I and j we know Ci ∩ CJ != ∅ (AKA every set overlaps some other set)**

5) Prove using an element argument that for all sets A and B,   
 (A ∩ B) ∪ (A ∩ Bc) = A.

**Say that an element x is in (A ∩ B) ∪ (A ∩ Bc) then this element is obviously in A. It is also in B or the complement of B. Since the union of B and its complement form the universal set, we see that x is in A and the universal set, which means x is in A if x is in (A ∩ B) ∪ (A ∩ Bc). So (A ∩ B) ∪ (A ∩ Bc) is a subset of A.**

**Next assume that an element x is in A. Then x will be in A ∩(U) where U is the universal. Now if we take any set B we know that U = (B ∪Bc), so x is in A ∩ (B ∪Bc). By DeMorgan’s laws we then know that x will be in (A ∩ B) ∪ (A ∩ Bc). So A is a subset of A ∩ B) ∪ (A ∩ Bc).**

**Since (A ∩ B) ∪ (A ∩ Bc) is a subset of A and A is a subset of (A ∩ B) ∪ (A ∩ Bc), we know A = (A ∩ B) ∪ (A ∩ Bc).**

6) For A = {1,2,3}, B = {2,3,4}, C = {3,4,5} find

(A X B) ∪ (A X C).

**A X B = {(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)}**

**A X C = {(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5)}**

**(A X B) ∪ (A X C) = {(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(1,5),(2,5),(3,5)}**

7) Given the relation on A = [1,2] and B = [1,2,4,5] defined by

{(1,1),(2,2),(1,5),(1,2),(2,4),(1,4)}

1. Find the inverse relation **{(1,1),(2,2),(5,1),(2,1),(4,2),(4,1)}**
2. Is this relation reflexive? **No (4,4) and (5,5) are not included**
3. Is this relation symmetric? **No (5,1) is not included**
4. Is this relation antisymmetric? **Yes a R b and b R a are only true when a = b**
5. Is this relation transitive? **Yes: Only example is (1,2), (2,4), (1,4) which works**
6. Is this relation an equivalence relation? **No, not reflexive or symmetric**
7. Is this relation a partial order relation? **No, not reflexive**
8. Could you make an equivalence relation using A and B? **Yes, take relation defined by {(1,1),(2,2)}**

8) Find numbers congruent to each of the following.

a) 500500 mod 5 **=== 0**

b) 392 mod 6 **=== 2**

c) 166 mod 3 **=== 1**

9) Give a surjection from the set of real numbers onto the set A = {2}. If this is not possible, say why. **Take F: R->A where F(x) = 2 for any real number x. This is a surjection since we cover all of A (you could also say our range = our co-domain).**

10) State whether the following functions are injective (one-to-one), surjective (onto), or both (bijective or one to one correspondence). Given an example if something is not injective or surjective.

a) Function F: R -> R F(x) = 2x **This is a bijection:**

**if F(x1) = F(x2) then 2x1 = 2x2 and x1 = x2 (therefore it is an injective function).**

**Also this function produces any real number (therefore it is surjective).**

b) Function G: R-> R G(x) = 2x when x != 1 or

3 when x = 1 **Neither**

**When x1 = 1.5 and x2 =1 F(x1) = F(x2) but x1 != x2 (Not an injection)**

**There exists no x such that F(x) = 2 (Not a surjection)**

c) Function H:N->R H(x) = 2x **Only injective**

**This is an injection for the same reason a is an injection**

**However if we take H(x) = 5/3 (or any other non-natural number or odd number) we will not be able to find an x to satisfy this equation.**

11) For g: R-> R g = sin(x) and f: R-> R f = π\*x, calculate the following.

a) g ◦ f(1/2) **= sin(pi/2) = 1**

b) f ◦ g(0) **= pi\*sin(0) = pi\*0 = 0**

12) Make a surjection from N->R or say why it is impossible. (Hint: What are the cardinalities of these two sets?)

**This is impossible as the cardinality of the set of natural numbers is less than the cardinality of the set of Real numbers.**

**Another true answer: This is impossible because N is countable and R is uncountable**

13) Solve these logarithm problems (without a calculator)

a) log381 **= 4**

b) ln e2 **= 2**